

# On the zero set of $G$ -equivariant maps

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November 11, 2008

## Abstract

Let  $G$  be a finite group acting on vector spaces  $V$  and  $W$  and consider a smooth  $G$ -equivariant mapping  $f : V \rightarrow W$ . This paper addresses the question of the zero set near a zero  $x$  of  $f$  with isotropy subgroup  $G$ . It is known from results of Bierstone and Field on  $G$ -transversality theory that the zero set in a neighborhood of  $x$  is a stratified set. The purpose of this paper is to partially determine the structure of the stratified set near  $x$  using only information from the representations  $V$  and  $W$ . We define an index  $s(\Sigma)$  for isotropy subgroups  $\Sigma$  of  $G$  which is the difference of the dimension of the fixed point subspace of  $\Sigma$  in  $V$  and  $W$ . Our main result states that if  $V$  contains a subspace  $G$ -isomorphic to  $W$ , then for every maximal isotropy subgroup  $\Sigma$  satisfying  $s(\Sigma) > s(G)$ , the zero set of  $f$  near  $x$  contains a smooth manifold of zeros with isotropy subgroup  $\Sigma$  of dimension  $s(\Sigma)$ . We also present a systematic method to study the zero sets for group representations  $V$  and  $W$  which do not satisfy the conditions of our main theorem. The paper contains many examples and raises several questions concerning the computation of zero sets of equivariant maps. These results have application to the bifurcation theory of  $G$ -reversible equivariant vector fields.

## 1 Introduction

The purpose of this paper is to introduce a new perspective and several new results for the study of  $G$ -equivariant maps  $f : V \rightarrow W$  where  $V$  and  $W$  are possibly non-isomorphic representations. The main goal of this paper is to investigate zero sets of such maps in a neighborhood of a zero with full isotropy subgroup  $G$ .

The widespread appearance of symmetry in differential equation models eventually led to the establishment of equivariant dynamical system as a subbranch of dynamical systems. One of the successful uses of equivariant dynamical systems is in the study of bifurcation problems and equivariant bifurcation theory of vector fields is now a standard tool for the study of bifurcations in symmetric differential equation models from all areas of science and engineering.

Fundamental results making up the foundations of the theory of local bifurcations of vector fields in the presence of symmetry can be found in the works of Michel [16], Ruelle [17] and Sattinger [18]. The use of singularity theory for studying local bifurcation problems of equivariant vector fields goes back to Golubitsky and Schaeffer [11] and a comprehensive treatment is found in Golubitsky, Stewart and Schaeffer [12]. The approach of [12] has been particularly successful in the study of bifurcation problems arising from mathematical models. Local bifurcation problems of equivariant vector fields have also been studied using  $G$ -transversality theory (a.k.a equivariant general position) and the results can be found in the work of Field and collaborators, see Field [6, 7] for details. For instance, the  $G$ -transversality approach led to a characterization of the criteria under which the so-called Maximum Isotropy Subgroup Conjecture (MISC) holds, see for instance [5].

For local bifurcations of  $G$ -equivariant bifurcation problems  $f(x, \lambda) = 0$  where  $x \in V$ ,  $\lambda \in \mathbb{R}^\ell$  and  $f : V \times \mathbb{R}^\ell \rightarrow V$  is  $G$ -equivariant, the starting point is the analysis of a zero of the  $G$ -equivariant mapping  $f$ . Note that in this case, the  $V$  in the domain and image are isomorphic  $G$ -representations and the inverse function theorem shows that equilibrium solutions are generically isolated. However, vector fields which are not only equivariant but also possess properties of time-reversibility, known as  $G$ -reversible equivariant vector fields, are described by smooth maps  $f : V \times \mathbb{R}^\ell \rightarrow V_\sigma$  where  $V_\sigma$  is a  $G$ -representation of  $V$  possibly non-isomorphic to  $V$  and determined by the type of time-reversibility. Recent progress on the steady-state bifurcation theory of  $G$ -reversible equivariant vector fields [2] shows that for large classes of these vector fields, equilibrium solutions are no longer isolated. To determine the zero set in a neighborhood of equilibrium solutions, one has to study  $G$ -equivariant maps from non-isomorphic  $G$ -spaces  $V$  and  $W$  where  $W$  is a subrepresentation of  $V_\sigma$ .

It is known from  $G$ -transversality theory that generic zero sets of general  $G$ -equivariant maps  $f : V \rightarrow W$  are Whitney regular stratified sets [1] and [4]. A stratified set is a locally finite collection of submanifolds and Whitney regularity is a technical condition on the way the submanifolds fit together. Several examples of zero sets are computed in the context of  $G$ -reversible equivariant systems, see [7] and [2] where the stratified structure of the zero is partially obtained. The insight gained from these examples shows that partial information about the zero set is encoded in the form of an index which is the difference of the dimensions of fixed point subspaces for isotropy subgroups of  $V$  and  $W$ .

In this paper, we explore this issue and show results which confirm the insight in several cases. We begin by proving in Theorem 1.1 a nonlinear version of Schur's lemma; that is, we give a sufficient condition on the representations  $V$  and  $W$  for a  $G$ -equivariant mapping  $f : V \rightarrow W$  to be identically zero. Our main theorem is the following: suppose  $G$  is a finite group acting on  $V$  and  $W$  where  $V$  contains a subrepresentation  $G$ -isomorphic to  $W$ . Let  $f : V \rightarrow W$  be a  $G$ -equivariant map such that  $f(0) = 0$ , then for each maximal isotropy subgroup  $\Sigma$  with index greater than the index of  $G$ , the zero set of  $f$  near 0 contains a submanifold of zeros with isotropy subgroup  $\Sigma$  of dimension given by the index. The proof of this result is obtained using a reduction of the problem to isotypic components of  $W$  and applying the implicit function theorem. Moreover, for the cases not treated using this result, we present a method suitable for explicit examples and which uses a result of Buchner et al [3]. This result requires only the computation of the lowest degree equivariants. This is a significant advantage to the alternative method which requires the computation of a minimal set of equivariant generators for smooth  $G$ -equivariant maps. This is a tedious task, often requiring the use of symbolic algebra packages.

Note that  $G$ -transversality theory has been formulated in the context of manifolds and that many of the results obtained in this paper can be lifted to smooth mappings between  $G$ -manifolds

using the Slice Theorem [7]. For instance,  $G$ -transversality is used in the study of low-dimensional manifolds supporting a group action. In fact, the number which corresponds to the index as defined in this paper appears in Hambleton [13], but it is not explicitly singled out.

The paper is organized as follows. In the first section, we state and prove Theorem 1.1 and then present some elementary examples which leads to the statement of our main result, Theorem 1.8. The following section discusses the relevance of these questions in the context of steady-state bifurcations of  $G$ -reversible equivariant vector fields. Section 1.1 contains all the prerequisites concerning local zero sets, stratifications and  $G$ -transversality. Section 4 presents known results about the dimension of zero sets with symmetry obtained from stratumwise transversality. Section 5 presents known and new results which enable us to reduce the calculations along isotypic components of the  $W$  representation. In section 6, we show Theorem 6.2 and this is the main ingredient in the proof of Theorem 1.8 also in this section. Then, we present the computational method based on a result of [3] to study the cases not covered by Theorem 6.2. More questions are listed in the final section.

## 1.1 Main Theorems

Let  $V$  and  $W$  be finite dimensional vector spaces. Let  $G$  be a compact Lie group and  $\rho_V : G \rightarrow \mathbf{O}(V)$ ,  $\rho_W : G \rightarrow \mathbf{O}(W)$  be two representations of  $G$ . Recall that if  $\rho$  is a representation then  $\ker \rho = \{g \in G \mid \rho(g) = I\}$  and a representation is faithful if  $\ker \rho = \{1\}$ .

Let  $f : V \rightarrow W$  be a smooth map commuting with the respective actions of  $G$  on  $V$  and  $W$ :

$$f(\rho_V(g)x) = \rho_W(g)f(x). \quad (1)$$

Then  $f$  is said to be  $G$ -equivariant and we denote this set of functions by  $C_G^\infty(V, W)$ . Let  $x \in V$ , the set

$$G_x = \{g \in G \mid gx = x\}$$

is a subgroup of  $G$  called the *isotropy subgroup* of  $x$ . Let  $(G_x)$  denote the conjugacy class of  $G_x$ , the conjugacy class of an isotropy subgroup is called the *isotropy type*. We write  $\iota(x)$  for the isotropy type of the point  $x$ . Denote by  $\mathcal{O}(V, G)$  the set of isotropy types for the action of  $G$  on  $V$ . One can define a partial order on this set by the following rule: let  $\tau, \mu \in \mathcal{O}(V, G)$ , then  $\tau > \mu$  if there exists  $H \in \tau$  and  $K \in \mu$  such that  $H \supsetneq K$ . To each isotropy subgroup  $\Sigma$  is associated a *fixed point subspace*

$$\text{Fix}(\Sigma) = \{x \in V \mid \sigma x = x \text{ for all } \sigma \in \Sigma\}$$

and for each isotropy type  $\tau$ , we define the *orbit stratum*  $V_\tau$  as

$$V_\tau = \{x \in V \mid \iota(x) = \tau\}.$$

An important feature of  $G$ -equivariant maps is that they preserve fixed point subspaces:

$$f : \text{Fix}_V(\Sigma) \rightarrow \text{Fix}_W(\Sigma).$$

The proof is straightforward as we now show. Let  $x \in \text{Fix}_V(\Sigma)$  and  $\sigma \in \Sigma$  then

$$f(x) = f(\rho_V(\sigma)x) = \rho_W(\sigma)f(x).$$

We now study the effects of the faithfulness of the representations  $V$  and  $W$  on the zero set of  $f$ . The next result is stated for an isotypic component  $W$ . This is not a restriction since the study of the zero set of  $f$  can be decomposed along isotypic components, more on that in section 5. Note that this next result can be interpreted as a nonlinear version of Schur's lemma and treats the case where the representation  $V$  is not faithful and  $\ker \rho_V \cap \ker \rho_W = \{1\}$ .

**Theorem 1.1** *Suppose that assumptions (5.2) and (5.5) are satisfied. Let  $(V, \rho_V)$  and  $(W, \rho_W)$  be  $G$ -spaces where  $W = U \oplus \cdots \oplus U$  is an isotypic component, and  $V$  does not contain an irreducible representation isomorphic to  $U$ . Let  $f : V \rightarrow W$  be a  $G$ -equivariant map. Suppose that  $\ker \rho_V \neq \{1\}$  and  $\ker \rho_W \cap \ker \rho_V = \{1\}$ . Then,  $f$  is identically zero.*

**Proof:** Suppose that  $W$  contains  $m$  copies of the irreducible representation  $U$ . Let  $f(x) = (f_1(x), \dots, f_m(x))$  where  $f_i : V \rightarrow U$  is also  $G$ -equivariant for  $i = 1, \dots, m$  since  $U$  is irreducible. Let  $\rho_V$  be the representation of  $G$  on  $V$  and  $\Sigma = \ker \rho_V \neq \{1\}$ . Then, for all  $\sigma \in \Sigma$

$$f_i(x) = f_i(\sigma x) = \sigma f_i(x)$$

which means that  $f_i(x) \in \text{Fix}_U(\Sigma)$  for all  $x \in V$ . But since  $f_i$  is  $G$ -equivariant, we have that  $\sigma f_i(gx) = f_i(gx)$  and so  $\sigma g f_i(x) = g f_i(x)$  implies

$$[g^{-1}\sigma g]f_i(x) = f_i(x)$$

for all  $x \in V$ . Thus,

$$f_i(x) \in \bigcap_{g \in G} \text{Fix}_U(g^{-1}\Sigma g)$$

for all  $x \in V$ . We define the subgroup

$$H := \langle g^{-1}\sigma g \mid g \in G, \sigma \in \Sigma \rangle.$$

Since  $g^{-1}\Sigma g$  is a subgroup of  $H$  for all  $g \in G$ , then  $f_i(x) \in \text{Fix}_U(H)$  for all  $x \in V$ .

For all  $k \in G \setminus H$  and  $y \in V$ , we have  $f_i(ky) \in \text{Fix}_U(H)$  and

$$f_i(ky) = k f_i(y) \in k \text{Fix}_U(H).$$

Thus

$$f_i(x) \in \bigcap_{k \in G} k \text{Fix}_U(H)$$

since for any  $x \in V$  there exists  $k \in G$  and  $y_k \in V$  such that  $x = ky_k$ . But,  $\bigcap_{k \in G} k \text{Fix}_U(H)$  is a  $G$ -invariant subspace of the irreducible representation  $U$ . Therefore,  $\bigcap_{k \in G} k \text{Fix}_U(H)$  is either  $U$  or  $\{0\}$ . Suppose that  $\bigcap_{k \in G} k \text{Fix}_U(H) = U$  then  $\text{Fix}_U(H) = U$  but this would mean that  $\ker \rho_W \supset H \supset \ker \rho_V \neq \{1\}$ , implying  $\ker \rho_V \cap \ker \rho_W \neq \{1\}$  which is a contradiction. Therefore,  $\bigcap_{k \in G} k \text{Fix}_U(H) = \{0\}$ , which implies that  $f_i \equiv 0$  for all  $i = 1, \dots, m$  and so  $f \equiv 0$ . ■

**Remark 1.2** Note that a more algebraic proof of this result can be obtained by showing that the dimension of  $G$ -equivariant maps of homogeneous degree  $d$  is zero for all  $d$ . By the trace formula, one can show that

$$\dim(\mathbb{C}[V]_d \otimes W)^G = (\chi_{S^d}, \chi_W) \quad (2)$$

where  $S^d$  is the  $d^{\text{th}}$ -symmetric tensor product of  $V$ ,  $\chi$  is the character of the representation and  $(\cdot, \cdot)$  is the inner product of characters. We claim that  $(\chi_{S^d}, \chi_W) = 0$ . Indeed, suppose that  $(\chi_V^d, \chi_U) \neq 0$ . Since there exists a nonidentity element  $g \in G$  such that  $gv = v$  for all  $v \in V$  then for all integers  $d \geq 0$  and for all  $w \in V^{\otimes d}$  we have  $gw = w$ . Now,  $(\chi_V^d, \chi_U) \neq 0$  implies  $\chi_U(g) = \chi_U(1)$  and we have a contradiction since  $g \notin \ker \rho_U$  by assumption.

From Theorem 1.1, when considering  $f^{-1}(0)$ , one only needs to consider the cases  $\ker \rho_V \subseteq \ker \rho_W$  and  $\ker \rho_W \subseteq \ker \rho_V$ . Suppose that  $\ker \rho_W \subsetneq \ker \rho_V$  and consider  $G' = G/\ker \rho_W$ . Then,  $G'$  acts faithfully on  $W$  and choose  $g \in \ker \rho_V \setminus \ker \rho_W$ . Then,  $\rho_V$  restricted to  $g\ker \rho_W$  is the identity matrix. So,  $G'$  does not act faithfully on  $V$  and we are in the situation described in Theorem 1.1. Suppose that  $\ker \rho_V \subseteq \ker \rho_W$ . Then,  $G' = G/\ker \rho_V$  acts faithfully on  $V$  and Theorem 1.1 does not apply in this case. The next assumption holds for the remainder of the paper.

**Assumption 1.3**  $\rho_V$  is a faithful representation.

Zero sets of general  $G$ -equivariant mappings with given isotropy subgroup  $\Sigma$  typically arise as families where the dimension is given by the difference in the dimensions of the fixed point subspaces in  $V$  and  $W$ . The following example illustrates this fact.

**Example 1.4** Let  $V = \mathbb{R}^3$  and  $G = \mathbb{Z}_2(R)$  act on  $V = \mathbb{R}^3$  by  $R.(x, y, z) = (x, y, -z)$  and  $W = \mathbb{R}^3$  as  $-R$ . This is an example of a mapping giving rise to a  $G$ -reversible vector field in  $V$ . The general form of the germ at zero is given by:

$$\dot{x} = p(x, y, z^2)z, \quad \dot{y} = q(x, y, z^2)z, \quad \dot{z} = r(x, y, z^2)$$

where  $p, q$  and  $r$  are smooth functions. The isotropy subgroup  $\mathbb{Z}_2$  is such that  $\dim \text{Fix}_V(\mathbb{Z}_2) = 2$  and  $\dim \text{Fix}_W(\Sigma) = 1$ . Suppose that  $(x_0, y_0, 0) \in \text{Fix}_V(\mathbb{Z}_2)$  is an equilibrium; that is,  $r(x_0, y_0, 0) = 0$ . Let

$$f(x, y, z) = \begin{bmatrix} p(x, y, z^2)z \\ q(x, y, z^2)z \\ r(x, y, z^2) \end{bmatrix}$$

and define

$$f_{\mathbb{Z}_2} = f|_{\text{Fix}_V(\mathbb{Z}_2)} : \text{Fix}_V(\mathbb{Z}_2) \rightarrow \text{Fix}_W(\mathbb{Z}_2).$$

In fact,

$$f_{\mathbb{Z}_2}(x, y, 0) = \begin{bmatrix} 0 \\ 0 \\ r(x, y, 0) \end{bmatrix}$$

and if the nondegeneracy assumption  $dr(x_0, y_0, 0) \neq 0$  is satisfied, then the implicit function theorem implies that  $(x_0, y_0, 0)$  is part of a one-dimensional family of equilibria with the same isotropy subgroup. Note that the dimension of the set of equilibria is the difference between the dimensions of the fixed point subspaces for  $\mathbb{Z}_2$  in  $V$  and  $W$  and this is a general fact which is encoded in the "index" definition below. Equilibria with trivial isotropy subgroup appear as isolated points and are therefore bounded away from  $\mathbb{Z}_2$  symmetric equilibria.

In Section 2, we discuss the results of this paper in the context of steady-state bifurcations of  $G$ -reversible equivariant problems.

**Definition 1.5** Let  $\Sigma$  be a subgroup of  $G$ . Then the *index* of  $\Sigma$  is defined as

$$s(\Sigma; V, W) = \dim \text{Fix}_V(\Sigma) - \dim \text{Fix}_W(\Sigma).$$

If there are no ambiguities about the spaces, we can write  $s(\Sigma)$ .

This index is defined in Field [7] and also in Buono et al [2] in the context of reversible-equivariant vector fields where it is called the  $\sigma$ -index where  $\sigma$  is the sign-representation of the group. It is a straightforward application of transversality theory that for finite groups, given an isotropy subgroup  $\Sigma$ , generically, zero sets of equilibria form a smooth  $s(\Sigma)$  dimensional manifold, see for instance Field [7] or section 4 below. Example 1.4 shows that zeros of  $f$  with isotropy subgroup  $\mathbb{Z}_2$  and 1 are bounded away from each other. We now turn to a simple example which illustrates the question of the embedding of zeros with a given isotropy subgroup  $\Sigma$  within zeros with isotropy subgroups  $\Sigma' \subset \Sigma$ .

**Example 1.6** Let  $\mathbb{D}_2$  be the group generated by the elements  $\kappa, \sigma$  which act on  $V = \mathbb{R}^2$  and  $W = \mathbb{R}$  as follows:

$$\begin{aligned} \kappa.(x, y) &= (x, -y), & \kappa.u &= u, \\ \sigma.(x, y) &= (-x, y), & \sigma.u &= -u. \end{aligned}$$

The isotropy subgroup lattice including the index is given in Figure 1. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a smooth

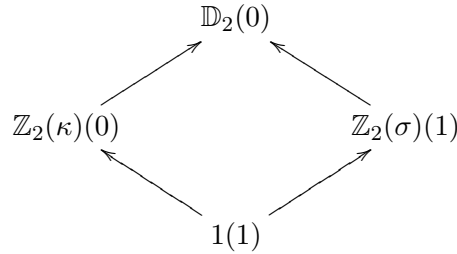


Figure 1: Isotropy subgroup lattice for the  $\mathbb{D}_2$  action on  $\mathbb{R}^2$

$\mathbb{D}_2$ -equivariant mapping. Since  $f : \text{Fix}_V(\Sigma) \rightarrow \text{Fix}_W(\Sigma)$  for all isotropy subgroups  $\Sigma$ , zeros of  $f$  with isotropy subgroup  $\Sigma$  generically appear in  $s(\Sigma)$ -dimensional families. From this consideration alone, we see that the zero at the origin with isotropy subgroup  $\mathbb{D}_2$  can be embedded within the family of zeros with isotropy subgroup  $\mathbb{Z}_2(\sigma)$  and 1, but would be isolated from zeros with isotropy subgroup  $\mathbb{Z}_2(\kappa)$ . Zeros with isotropy subgroup  $\mathbb{Z}_2(\kappa)$  could be embedded inside a family of zeros with trivial isotropy subgroup. We now investigate these options.

The general smooth  $\mathbb{D}_2$ -equivariant map is

$$f(x, y) = p(x^2, y^2)x.$$

Therefore  $f = 0$  if and only if  $p(x^2, y^2) = 0$  or  $x = 0$ .  $\text{Fix}(\mathbb{Z}_2(\sigma)) = \{(x, y) \mid x = 0\}$  is a subset of  $f^{-1}(0)$  and so the origin is contained inside this one-dimensional family. A necessary condition for the origin to be embedded within a family of zeros with trivial isotropy subgroup is that it satisfies the (non-generic) condition  $p(0, 0) = 0$ . Figure 1.6(a) shows this situation. Therefore, the generic situation in this case is that the origin is embedded within a family of zeros with isotropy subgroup  $\mathbb{Z}_2(\sigma)$ , but not with trivial isotropy subgroup.

Suppose that  $p(x_0^2, 0) = 0$  for some  $x_0 \neq 0$ . The zero at  $(x_0, 0)$  is included in a family of zeros with trivial isotropy subgroup if a nondegeneracy condition on the derivatives of  $p(x^2, y^2)$  is satisfied. This generic situation is illustrated in Figure 1.6(b).

Given this example, we can now state our first question.

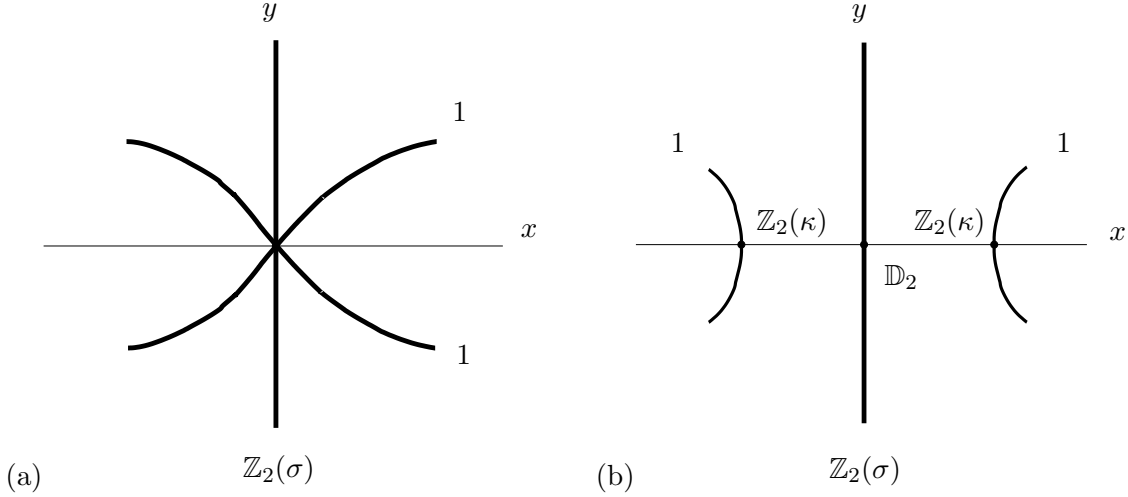


Figure 2: Zero set of  $\mathbb{D}_2$  equivariant map  $f$ : (a) non-generic case (b) generic case with  $\mathbb{Z}_2(\kappa)$  equilibria

**Question 1.7** Let  $G$  be a finite group and  $\Sigma', \Sigma$  be two isotropy subgroups of  $G$ . If  $\Sigma'$  is a maximal isotropy subgroups of  $\Sigma$  and  $s(\Sigma') > s(\Sigma)$ , is it true that generically the zero set with isotropy subgroup  $\Sigma$  is embedded within the zero set with isotropy subgroup  $\Sigma'$ ?

In this paper, we give a partial answer to this question in the case where  $\Sigma = G$ . Recall that two representations  $V$  and  $W$  are  $G$ -isomorphic if there exists a nonzero linear  $G$ -equivariant mapping  $A : V \rightarrow W$ . This is the main result of the paper.

**Theorem 1.8** *Suppose that assumption 1.3 is satisfied. Let  $f : V \rightarrow W$  be a smooth  $G$ -equivariant map and suppose that  $V$  contains a subrepresentation which is  $G$ -isomorphic to  $W$ . Suppose that  $f$  is  $G$ -transverse to  $0 \in W$  at  $0 \in V$  and let  $\Sigma$  be a maximal isotropy subgroup of  $G$  such that  $s(\Sigma) > s(G)$ . Then,  $f^{-1}(0)$  in a neighborhood of  $0 \in V$  contains a submanifold of points with isotropy subgroup  $\Sigma$  of dimension  $s(\Sigma)$  containing  $0 \in V$ .*

$G$ -transversality is defined in section 3 and is required in this result as a nondegeneracy condition. The proof of Theorem 1.8 is a consequence of the implicit function theorem and it is found in section 6. Now, there are many examples where the condition on  $V$  and  $W$  in the above theorem does not hold and these are studied in section 6.2 and section 6.3. There, we present an explicit computational method to determine the structure of the zero set near the origin for representations  $V$  and  $W$  which do not satisfy the conditions of Theorem 1.8. Moreover, this method only requires the computation of homogeneous equivariant generators of lowest degree. In section 7, we list some more questions concerning local zero sets of  $G$ -equivariant mappings.

## 2 Application: reversible-equivariant vector fields

The theory developed in this paper has immediate application to the steady-state bifurcation theory of  $G$ -reversible equivariant vector fields. It is a generic feature of reversible-equivariant systems that equilibrium solutions appear as nontrivial stratified sets.

In [2], a systematic study of steady-state bifurcations in smooth  $G$ -reversible equivariant vector fields is presented.  $G$ -reversible equivariant vector fields are defined as follow. Let  $V$  be a vector space and consider the representations

$$\begin{aligned}\rho &: G \rightarrow \mathbf{O}(V) \\ \sigma &: G \rightarrow \mathbb{Z}_2 = \{+1, -1\} \\ \rho_\sigma &: G \rightarrow \mathbf{O}(V); \quad \rho_\sigma(g) = \sigma(g)\rho(g).\end{aligned}\tag{3}$$

A mapping  $f$  is  $G$ -reversible equivariant if for all  $g \in G$

$$f(\rho(g)x, \lambda) = \rho_\sigma(g)f(x, \lambda).\tag{4}$$

Then, the  $\ell$ -parameter family of dynamical systems

$$\frac{dx}{dt} = f(x, \lambda),\tag{5}$$

where  $\lambda \in \mathbb{R}^\ell$  is an  $\ell$ -dimensional parameter vector is  $G$ -reversible equivariant.

Let  $L : V \rightarrow V$  be a linear  $G$ -reversible equivariant map commuting with the representations  $\rho$  and  $\rho_\sigma$  of  $G$  on  $V$ . If  $\rho$  and  $\rho_\sigma$  are nonisomorphic irreducible representations, Schur's lemma implies  $L \equiv 0$  and  $L$  is forced to have a kernel by the representations. For general representations  $\rho$  and  $\rho_\sigma$ , the map  $L$  has a nontrivial kernel forced by the group representations if the isotypic decompositions of the representations  $\rho$  and  $\rho_\sigma$  are not isomorphic. The *forced kernel* for a linear reversible equivariant map  $L$  is the isomorphism class (as a group representation) of the lowest dimensional kernel that a reversible equivariant map can have between these representations.

We decompose  $V = W_1 \oplus W_2$  where  $W_2$  is the forced kernel and  $W_1$  is an orthogonal complement of  $W_2$ . By construction,  $W_2$  and  $W_{2,\sigma}$  have no common irreducible representations and  $W_1$  is isomorphic to  $W_{1,\sigma}$ . In particular, there exists an invertible  $G$ -reversible equivariant linear mapping  $T : W_1 \rightarrow W_{1,\sigma}$ . See [2] for a complete characterization of forced kernels in terms of irreducible representations of  $G$ .

Thus, a  $G$ -reversible equivariant mapping  $f : V \rightarrow V_\sigma$  can be decomposed along  $W_1$  and  $W_2$  as

$$f_1 : V \rightarrow W_1, \quad f_2 : V \rightarrow W_2.$$

Suppose that  $f(0) = 0$  and we want to characterize the zero set in the neighborhood of  $x = 0$ . The simplest case arises if  $d_{W_1}f_1(0)$  is nonsingular, then by the implicit function theorem, there exists a smooth  $G$ -reversible equivariant mapping  $\phi : W_2 \rightarrow W_1$  with  $\phi(0) = 0$  such that in a neighborhood of  $w_2 = 0$ ,

$$f_1(\phi(w_2), w_2) \equiv 0.$$

Therefore, the zero set of  $f$  near  $x = 0$  is characterized by  $\tilde{f}_2 : W_2 \rightarrow W_{2,\sigma}$  where  $\tilde{f}_2(w_2) = f_2(\phi(w_2), w_2)$  where  $W_2$  and  $W_{2,\sigma}$  do not share any irreducible representations. Note that our Theorem 1.8 does not hold in this situation and so the methods presented in sections 6.2 and 6.3 are relevant. We now return to Example 1.4 to illustrate this general construction.

**Example 2.1** The action of  $R$  on  $V = \mathbb{R}^3$  has two copies of the trivial representation  $T$  and one copy of the  $-1$  representation  $A$ . Let  $W_1 = T \oplus A = \{(x, y, z) \mid x = 0\}$  and  $W_2 = T = \{(x, y, z) \mid y = z = 0\}$ . The mapping  $f$  decomposes as  $f_1(x, y, z) = (q(x, y, z^2)z, r(x, y, z^2))$  and  $f_2(x, y, z) = p(x, y, z^2)z$ . If  $f(0) = 0$  then  $r(0, 0, 0) = 0$  and  $d_{W_1}f_1(0, 0, 0)$  is nonsingular, then  $(y, z) = \phi(x) := (\phi_1(x), \phi_2(x))$  where  $\phi : W_1 \rightarrow W_2$  is  $R$ -equivariant, smooth and  $\phi(0) = (0, 0)$ . Then,  $\tilde{f}_2 : W_2 \rightarrow W_{2,\sigma}$  and since  $R$  acts trivially on  $W_2$  then  $\tilde{f}_2(x) = -\tilde{f}_2(x)$ . That is,  $\tilde{f}_2 \equiv 0$  and this defines a smooth one-dimensional family of zeros of  $f$ .



At bifurcation points, Buono et al [2] have shown that  $G$ -reversible equivariant bifurcation problems near "organizing centres" can be reduced to  $G$ -equivariant bifurcation problems with possible parameter symmetry. Bifurcation problems with parameter symmetry have been studied by [8] and other authors and these bifurcation problems can be analyzed also in the context provided by this paper.

Indeed, consider a smooth vector field  $\dot{x} = f(x, \lambda)$  where  $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^p$  and suppose that  $G$  acts nontrivially on  $\mathbb{R}^n$  and the parameters  $\mathbb{R}^p$ . Then,  $G$ -equivariance of the vector field means that  $f : V \rightarrow W$  where  $V = \mathbb{R}^n \times \mathbb{R}^p$  and  $W = \mathbb{R}^n$ . See Furter *et al* [8] for more on bifurcation problems with parameter symmetries.

### 3 Zero sets and stratifications

In this section, we introduce concepts and notations in order to study zero sets of  $G$ -equivariant maps. A good reference for these results is the recent book by Field [7]. We discuss briefly the results from stratification theory which are needed for our purposes and recall the definition of  $G$ -transversality.

**Local zero sets** The local structure of  $f^{-1}(0)$  is investigated in the following way. The set  $C_G^\infty(V, W)$  is a finitely generated module over the ring of  $G$ -invariant maps  $C^\infty(V)^G$ . A minimal set of homogeneous generators (MSG)  $\{F_1, \dots, F_k\}$  has the property that all  $f \in C_G^\infty(V, W)$  can be written as

$$f = \sum_{i=1}^k h_i F_i,$$

where  $h_i : V \rightarrow \mathbb{R}$ ,  $i = 1, \dots, k$  are  $G$ -invariant smooth maps. We define the map  $F : V \times \mathbb{R}^k \rightarrow W$  by

$$F(x, t) = \sum_{i=1}^k t_i F_i(x). \quad (6)$$

The set  $\mathcal{E} = F^{-1}(0) \subset V \times \mathbb{R}^k$  contains the information about all possible zero sets for elements  $f \in C_G^\infty(V, W)$ . We refer to  $\mathcal{E}$  as the *universal variety* or also the *universal zero set*. The *graph map* of  $f$  is a map  $\text{graph}_f : V \rightarrow V \times \mathbb{R}^k$  defined by

$$\text{graph}_f(x) = (x, h_1(x), \dots, h_k(x)). \quad (7)$$

It is easy to see that

$$f = F \circ \text{graph}_f$$

and  $f^{-1}(0) = \text{graph}_f^{-1}(\mathcal{E})$ .

**Stratifications** We now review some basic facts about stratifications of sets. Gibson *et al* [9] is a good reference for the results stated below and more on stratifications.

A *stratification*  $S$  of a subset  $X$  of  $\mathbb{R}^n$  is a locally finite partition of  $X$  into smooth and connected submanifolds of  $\mathbb{R}^n$  called *strata*. In particular, if  $X$  is semialgebraic, then  $S$  is a semialgebraic stratification if each stratum is semialgebraic. A stratification is said to be *Whitney regular* if it satisfies a Whitney regularity condition. The details of Whitney regularity are not needed in our work and we refer the interested reader to [9]. Of particular interest to us is the *frontier condition*

of Whitney regular stratifications: if  $A, B$  are strata satisfying  $\overline{A} \cap B \neq \emptyset$  then  $B \subset \overline{A}$  or also  $B \subset \partial A$ . Another important consequence of Whitney regularity in our context is the following. If  $\overline{A} \cap B \neq \emptyset$ , then any map transverse to  $B$  is also transverse to  $A$  near  $B$ . Denote the union of  $i$ -dimensional strata of  $S$  by  $S_i$ ,  $i \geq 0$ . Then the following property holds.

**3.1** If  $S$  is a Whitney regular stratification, then  $S_i$  does not meet  $\overline{S}_j$  unless  $i \leq j$ .

Now, any semialgebraic set admits a canonical Whitney stratification having finitely many semialgebraic strata. In particular, let  $E$  be a subset of a smooth manifold  $X$ . A point  $x \in E$  is *regular* (of dimension  $d$ ) if  $x$  has a neighborhood  $F$  in  $X$  such that  $E \cap F$  is a smooth submanifold of (dimension  $d$ ). Every nonempty semialgebraic set has at least one regular point, in fact the regular points of maximal dimension are open and dense.

The local structure of zero sets of  $G$ -equivariant maps has been determined independently by Bierstone [1] and Field [4] in their work on “equivariant general position” equivalently “ $G$ -transversality”. Let  $P$  be a subrepresentation of  $W$ , then we say that  $f$  is  $G$ -transverse to  $P$  at some point  $x$  if  $\text{graph}_f(x)$  is transverse to the Whitney regular stratification of  $\mathcal{E}$ . In particular, they show the typical properties expected for transversality theorems; that is, the set

$$\{f \in C_G^\infty \mid f \text{ is } G\text{-transverse to } P\}$$

is open and dense in  $C_G^\infty(V, W)$  and an isotopy theorem also holds.

From standard stratification theory one obtains that the structure of  $f^{-1}(P)$  in a neighborhood of a point  $x \in f^{-1}(P)$  is an algebraic set which admits a  $G$ -invariant Whitney regular stratification. It is sufficient to study the case  $P = \{0\}$  since if  $P'$  is a  $G$ -invariant complement to  $P$  in  $W$  and  $\pi : W \rightarrow P'$  is the associated projection, then  $f^{-1}(P) = (\pi \circ f)^{-1}(0)$ .  $G$ -transversality of a mapping  $f$  to  $0 \in W$  at  $0 \in V$  is denoted by  $f \pitchfork_G 0 \in W$  at  $0 \in V$ .

Hence, the zero set of  $f$  is given by the intersection of  $\text{graph}_f$  with the universal variety  $\mathcal{E}$ . The set  $\mathcal{E}$  has a canonical Whitney regular stratification, thus  $\text{graph}_f^{-1}(\mathcal{E})$  also has a Whitney regular stratification, see [9].

## 4 Dimension of symmetric zero sets

In this section, we reformulate the problem of the zero set in a neighborhood of the origin in terms of inclusion of stratum in the boundary of larger strata. Consider the sets

$$\mathcal{E}_\Sigma = \{(x, t) \in \mathcal{E} \mid x \in V_\Sigma\}$$

over all isotropy types  $(\Sigma)$ . These are semialgebraic sets and  $\mathcal{E}$  is the disjoint union of  $\mathcal{E}_\Sigma$  over all isotropy types  $(\Sigma)$ . We denote the union of strata of regular points (of maximal dimension) in  $\mathcal{E}_\Sigma$  by  $R_\Sigma$ . Note that  $R_G$  is open and dense in  $\mathcal{E}_G$ . For an isotropy type  $(\Sigma)$ , a standard notation is  $n_\Sigma = \dim N_G(\Sigma)/\Sigma$ . The next result summarizes important properties of  $\mathcal{E}_\Sigma$ .

**Proposition 4.1 (Field [7] Lemma 6.9.2)** *Let  $(\Sigma)$  be an isotropy type for the action of  $G$  and  $S$  the Whitney regular stratification of  $\mathcal{E}$ . Then,*

(1)  $\mathcal{E}_\Sigma$  is a semialgebraic submanifold of  $V \times \mathbb{R}^k$  of dimension

$$s(\Sigma; V, W) + \dim(G/\Sigma) - n_\Sigma + k, \text{ and}$$

(2)  $\mathcal{E}_\Sigma$  inherits a semialgebraic canonical Whitney stratification  $S_\Sigma$  from  $S$ .

For finite groups, the dimension of  $\mathcal{E}_\Sigma$  reduces to  $s(\Sigma; V, W) + k$ . The intersection property of the sets  $\mathcal{E}_\Sigma$  are given by the next result.

**Proposition 4.2 (Field [7] Lemma 6.9.1)** *Let  $(\Sigma'), (\Sigma)$  be isotropy types of  $G$ . Then  $(\Sigma') > (\Sigma)$  if and only if  $\mathcal{E}_{\Sigma'} \cap \partial\mathcal{E}_\Sigma \neq \emptyset$ .*

The previous result show that if  $(\Sigma') > (\Sigma)$  then some stratum of  $\mathcal{E}_{\Sigma'}$  is contained in  $\partial\mathcal{E}_\Sigma$ . However, to have  $\mathcal{E}_{\Sigma'} \subset \partial\mathcal{E}_\Sigma$  we must show that  $R_{\Sigma'}$  is contained in  $\partial\mathcal{E}_\Sigma$ . The next result shows a negative criterion for the inclusion of  $R_{\Sigma'}$  into  $\partial\mathcal{E}_\Sigma$ .

**Proposition 4.3 (Field [7] Lemma 6.9.3)** *Suppose that  $(\Sigma) < (\Sigma')$ . If*

$$s(\Sigma'; V, W) - n_{\Sigma'} \geq s(\Sigma; V, W) - n_\Sigma$$

*then  $\dim [\mathcal{E}_{\Sigma'} \cap \overline{\mathcal{E}_\Sigma}] < \dim \mathcal{E}_{\Sigma'}$ .*

We now turn to the question of stratumwise transversality and compute the dimensions of zero sets of  $G$ -equivariant maps. The next two results are straightforward consequences of transversality theory, but we state and prove them explicitly.

**Lemma 4.4** *Let  $f : V \rightarrow W$  be  $G$ -equivariant and suppose that  $f(x) = 0$ , where  $G_x = \Sigma$ . Then  $f \pitchfork_G 0$  at  $x$  implies that  $s(\Sigma; V, W) - n_\Sigma \geq 0$ . Moreover, if  $s(\Sigma; V, W) - n_\Sigma > 0$  then the converse is also true.*

**Proof:** Since  $f \pitchfork_G 0$  at  $x$ , then  $\text{graph}_f \pitchfork \mathcal{E}_\Sigma$  at  $x$ . Now,  $\mathcal{E}_\Sigma \subset \mathcal{X} = \{G(x, t) | (x, t) \in \text{Fix}_V(\Sigma) \times \mathbb{R}^k\}$ . So,  $\text{graph}_f : \text{Fix}_V(\Sigma) \rightarrow \mathcal{X}$  is such that

$$d(\text{graph}_f)_x(\text{Fix}_V(\Sigma)) + T_{\text{graph}_f(x)}\mathcal{E}_\Sigma = T_{\text{graph}_f(x)}\mathcal{X}.$$

Thus,

$$\dim d(\text{graph}_f)_x(\text{Fix}_V(\Sigma)) + \dim T_{\text{graph}_f(x)}\mathcal{E}_\Sigma \geq \dim T_{\text{graph}_f(x)}\mathcal{X}. \quad (8)$$

Since  $\dim d(\text{graph}_f)_x(\text{Fix}_V(\Sigma)) = \dim \text{Fix}_V(\Sigma)$ ,  $\dim T_{\text{graph}_f(x)}\mathcal{E}_\Sigma = s(\Sigma; V, W) - n_\Sigma + \dim G/\Sigma$  by Proposition 4.1 and  $\dim T_{\text{graph}_f(x)}\mathcal{X} = \dim \text{Fix}_V(\Sigma) + \dim G/\Sigma + k$ , then (8) simplifies to

$$s(\Sigma; V, W) - n_\Sigma \geq 0.$$

Now, if  $s(\Sigma; V, W) - n_\Sigma > 0$  then the left hand side of (8) is always greater than the right hand side so transversality is automatic. ■

**Proposition 4.5** *Suppose that  $\text{graph}_f$  has nontrivial transverse intersection with  $\mathcal{E}_\Sigma$ . Then*

$$\dim \text{graph}_f^{-1}(\mathcal{E}_\Sigma) = s(\Sigma) + \dim(G/\Sigma) - n_\Sigma.$$

*In particular, if  $G$  is finite then  $\dim \text{graph}_f^{-1}(\mathcal{E}_\Sigma) = s(\Sigma)$ .*

**Proof:** Note that  $\dim(d\text{graph}_f)_x(V) = \dim V$ . Since  $\text{graph}_f$  has nontrivial transverse intersection with  $\mathcal{E}_\Sigma$ , then

$$\dim(d\text{graph}_f)_x(V) + \dim T_{\text{graph}_f(x)}\mathcal{E}_\Sigma \geq \dim V + k. \quad (9)$$

which reduces to the dimension of the intersection  $s(\Sigma) + \dim G/\Sigma - n_\Sigma \geq 0$ .  $\blacksquare$

**Remarks 4.6** Note that a necessary condition for the intersection of  $\text{graph}_f$  with  $\mathcal{E}_\Sigma$  to be generically nontrivial is that

$$s(\Sigma) - n_\Sigma \geq 0. \quad (10)$$

## 5 Reduction along isotypic components of $W$

We begin by a series of results about the structure of the  $G$ -spaces  $V$  and  $W$  which lead to a simplification of the context in which we study the local zero set near the origin. These results can be found in Chapter 6 sections 6.6 and 6.7 of Field [7], but we include them here for completeness.

Set  $p = \dim \text{Fix}_V(G)$  and  $q = \dim \text{Fix}_W(G)$ . Let  $V = \text{Fix}_V(G) \oplus V'$ ,  $W = \text{Fix}_W(G) \oplus W'$  where  $V'$  and  $W'$  are the sum of the remaining isotypic components. Let  $\{F_1, \dots, F_\ell\}$  be a MSG for  $C_G^\infty(V, W)$ . Let  $\{e_1, \dots, e_q\}$  be the canonical basis of  $\text{Fix}_W(G)$ . Then we can set  $F_j = e_j$ ,  $j = 1, \dots, q$  and so  $F(x, t) = \sum_{i=1}^q t_i e_i + \sum_{i=q+1}^\ell t_i F_i(x)$ . Therefore  $(x, t) \in \mathcal{E}$  if and only if  $t_j = 0$  for  $j = 1, \dots, q$  and  $\sum_{i=q+1}^\ell t_i F_i(x) = 0$ . Write  $x = (x_0, x')$  where  $x_0 \in \text{Fix}_V(G)$  and  $x' \in V'$ . The coordinates  $x_0$  are  $G$ -invariant so that  $\sum_{i=q+1}^\ell t_i F_i(x) = \sum_{i=q+1}^\ell t_i F_i(x')$ . Consider  $C_G^\infty(V', W')$  with universal zero set  $\tilde{\mathcal{E}}$ . We have shown the following correspondence.

**Proposition 5.1**  $(x', t_{q+1}, \dots, t_\ell) \in \tilde{\mathcal{E}}$  if and only if  $(x_0, x', 0, \dots, 0, t_{q+1}, \dots, t_\ell) \in \mathcal{E}$ .

Thus we make the following assumption.

**Assumption 5.2**  $V$  and  $W$  contain no trivial representations.

The next two results shows that we can decompose the zero set problem along isotypic components of the  $W$  representation.

**Proposition 5.3 (Field [7] Lemma 6.6.9)** Let  $W = W_1 \oplus \dots \oplus W_k$  be the isotypic decomposition of  $W$ . Then

$$\{G_1^1, \dots, G_{\ell_1}^1, G_1^2, \dots, G_{\ell_2}^2, \dots, G_1^k, \dots, G_{\ell_k}^k\}$$

is a MSG for  $C_G^\infty(V, W)$  if and only if for all  $i = 1, \dots, k$ ,  $\{G_1^i, \dots, G_{\ell_i}^i\}$  is a MSG for  $C_G^\infty(V, W_i)$ .

Consider the  $G$ -equivariant mappings  $F^i : V \rightarrow W_i$  defined by

$$F^i(x, t^i) = \sum_{j=1}^{\ell_i} t_j^i G_j^i$$

where  $\{G_1^i, \dots, G_{\ell_i}^i\}$  is a MSG for  $C_G^\infty(V, W_i)$  and let

$$\mathcal{E}_\Sigma^i = \{(x, t) \in V_\Sigma \times \mathbb{R}^{\ell_i} \mid F^i(x, t) = 0\}.$$

The following proposition shows that inclusion of strata can be obtained by reducing the inclusion problem to each isotypic component of  $W$ .

**Proposition 5.4**  $\mathcal{E}_G^i \subset \partial\mathcal{E}_\Sigma^i$  for all  $i = 1, \dots, k$  if and only if  $\mathcal{E}_G \subset \partial\mathcal{E}_\Sigma$ .

**Proof:**  $\Leftarrow$ ) Suppose that  $\mathcal{E}_G \subset \partial\mathcal{E}_\Sigma$ . For all  $t \in \mathbb{R}^l$ , there exists a sequence  $(x^n, s^n) \in \mathcal{E}_\Sigma$  such that  $(x^n, s^n) \rightarrow (0, t)$ . Write  $t = (t_1, \dots, t_k)$  where  $t_i \in \mathbb{R}^{\ell_i}$  and for all  $n \in \mathbb{N}$ ,  $s^n = (s_1^n, \dots, s_k^n)$  where  $s_i^n \in \mathbb{R}^{\ell_i}$ . By definition of  $F^i$ ,  $(x^n, s_i^n) \in \mathcal{E}_\Sigma^i$  for all  $n$ . Hence for any  $t_i \in \mathbb{R}^{\ell_i}$ , there exists  $(x^n, s_i^n) \rightarrow (0, t_i)$ ; that is,  $\mathcal{E}_G^i \subset \partial\mathcal{E}_\Sigma^i$ .

$\Rightarrow$ ) Suppose that  $\mathcal{E}_G^i \subset \partial\mathcal{E}_\Sigma^i$  for all  $i$ . Choose any  $t \in \mathbb{R}^l$ ,  $t = (t_1, \dots, t_k)$  where  $t_i \in \mathbb{R}^{\ell_i}$  and let  $s^n = (s_1^n, \dots, s_k^n)$  where  $s_i^n \in \mathbb{R}^{\ell_i}$  and  $(x^n, s_i^n) \in \mathcal{E}_\Sigma^i$ . For all  $n$ ,  $F(x^n, s^n) = F^1(x^n, s_1^n) + \dots + F^k(x^n, s_k^n) = 0$ . Thus  $(x^n, s^n) \in \mathcal{E}_\Sigma$  and  $(x^n, s^n) \rightarrow (0, t)$ . ■

For the remainder of the paper we make the next assumption.

**Assumption 5.5**  $W$  is the direct sum of  $r$  irreducible representations isomorphic to  $U$ .

In the next section, we show the main results concerning the inclusion of  $\mathcal{E}_G$  inside  $\partial\mathcal{E}_\Sigma$ .

## 6 Inclusions of $\mathcal{E}_G$

As mentioned above, we assume for the remainder of the paper that Assumptions 5.2 and 5.5 are in force. The MSG for  $C_G^\infty(V, W)$  is taken to be  $\{F_1, \dots, F_\ell\}$ . The following remark summarizes the approach taken to obtain the proofs of the theorems.

**Remark 6.1** Recall that  $R_G$  is the subset of regular points (of maximal dimension) of  $\mathcal{E}_G$  and that  $R_G$  is open and dense in  $\mathcal{E}_G$ . Therefore, the inclusion  $\mathcal{E}_G \subset \partial\mathcal{E}_\Sigma$  is equivalent to  $R_G \subset \partial\mathcal{E}_\Sigma$ . Now, since  $\mathcal{E}$  has a Whitney regular stratification, by the frontier condition, to prove the inclusion  $R_G \subset \partial\mathcal{E}_\Sigma$  it is sufficient to show that there exists one element  $(0, t) \in R_G$  and a sequence  $\{(x_n, t^n)\} \subset \mathcal{E}_\Sigma$  such that  $(x_n, t^n)$  converges to  $(0, t)$ .

It is straightforward that if  $\Sigma$  is a maximal isotropy subgroup of  $G$  with  $s(\Sigma) > 0$  and  $\text{Fix}_W(\Sigma) = \{0\}$  then  $\mathcal{E}_G \subset \partial\mathcal{E}_\Sigma$ . We suppose hereafter that  $\text{Fix}_W(\Sigma) \neq \{0\}$ . Suppose that  $W$  is the direct sum of  $r$  irreducible representations  $U$  and let  $\delta := (\chi_V, \chi_U)$  where  $\chi_V$  and  $\chi_U$  are the characters of  $V$  and  $U$ . Three cases have to be considered:

- (1)  $\delta \geq r$
- (2)  $\delta = 0$ , and
- (3)  $0 < \delta < r$ .

The first case can be solved completely and is the topic of the next section. The second and third cases are not as straightforward and we do not present a general theorem. However, we show a systematic method to study particular examples in those cases.

### 6.1 First Case

This case is the only one which is amenable completely to the implicit function theorem. Therefore, we obtain a general result on the inclusion of  $\mathcal{E}_G$  into the boundary of neighbouring sets  $\mathcal{E}_\Sigma$ .

**Theorem 6.2** Let  $\Sigma$  be a maximal isotropy subgroup of  $G$  and suppose that  $s(\Sigma) > 0$ . If  $\delta \geq r$ , then  $\mathcal{E}_G \subset \partial\mathcal{E}_\Sigma$

**Proof:** By Schur's lemma, if  $L : V \rightarrow W$  is a linear map, then

$$L = [A, 0]$$

where  $A : \delta U \rightarrow W$  is

$$A = \begin{bmatrix} a_{11}I & \cdots & a_{1\delta}I \\ \vdots & \ddots & \vdots \\ a_{r1}I & \cdots & a_{r\delta}I \end{bmatrix}$$

for  $a_{ij} \in \mathbf{k}$ ,  $\mathbf{k} \in \mathbb{R}, \mathbb{C}$ .

By the above, there are  $r\delta$  linear generators in the MSG for  $C_G^\infty(V, W)$ . Let

$$L_0 = d_x F(0, t) = \sum_{i=1}^{r\delta} t_i L_i.$$

The matrix  $(a_{ij})$  is generically of rank  $r$ . Hence there exists  $\tau \in R_G$  such that for  $(\tau_1, \dots, \tau_{r\delta})$ ,  $L_0$  is of rank:  $r \times \dim U$ .

Let  $V = rU \oplus V'$  where  $V'$  is a  $G$ -invariant complement of a subspace which is the direct sum of  $r$  copies of  $U$ . By the implicit function theorem, there exists open neighborhoods  $N_1$  of  $0 \in V'$  and  $N_2$  of  $0 \in rU$  and a smooth  $G$ -equivariant mapping  $\phi : N_1 \rightarrow N_2$  such that

$$F(\phi(v'), v', \tau) \equiv 0.$$

Let  $\Sigma$  be a maximal isotropy subgroup of  $G$  with  $s(\Sigma) > 0$ . Then,  $\dim \text{Fix}_{V'}(\Sigma) = s(\Sigma)$  and  $(v', \phi(v'))$  for  $v' \in N_1$  defines a smooth  $s(\Sigma)$ -dimensional manifold through the origin in  $\text{Fix}_V(\Sigma)$ . Therefore,  $\mathcal{E}_G \subset \partial \mathcal{E}_\Sigma$ .  $\blacksquare$

The next example illustrates the use of Theorem 6.2. It is not standard in the equivariant bifurcation theory literature, so we present it in details. The group under study contains the smallest group in the family  $F_{p,q}$  with  $p$  prime and  $q \mid p-1$  for which there are non-isomorphic representations with maximal isotropy subgroups having dimension two; in fact, there are three such non-isomorphic representations. See [14] for a description of the family  $F_{p,q}$ .

**Example 6.3** Consider the group  $\Gamma = F_{13,4} \times \mathbb{Z}_2$ , where

$$F_{13,4} = \langle a, b : a^{13} = b^4 = 1, b^{-1}ab = a^5 \rangle.$$

$F_{13,4}$  has order 52, so  $\Gamma$  has order 104. Let  $\omega = e^{\frac{2\pi i}{13}}$ , and

$$\alpha = \omega + \omega^5 + \omega^8 + \omega^{12}, \quad \beta = \omega^2 + \omega^3 + \omega^{10} + \omega^{11}, \quad \gamma = \omega^4 + \omega^6 + \omega^7 + \omega^9.$$

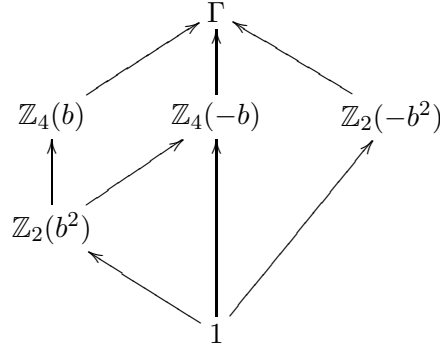
The character table in Figure 1 shows the characters of the three irreducible representations of degree 4, the full character table of  $F_{13,4}$  is in [14]. These three irreducible representations are denoted  $V_1, V_2$  and  $V_3$ . For  $\mu, \nu \in V_i$ ,  $i \in \{1, 2, 3\}$ ,

$$b.(\mu, \nu) = (\overline{\nu}, \mu)$$

and for the  $a$ -actions we have for  $(z_1, z_2) \in V_1$ ,  $(z_3, z_4) \in V_2$  and  $(z_5, z_6) \in V_3$

$$a.(z_1, z_2) = (\omega z_1, \omega^5 z_2) \quad a.(z_3, z_4) = (\omega^{10} z_3, \omega^{11} z_4) \quad \text{and} \quad a.(z_5, z_6) = (\omega^7 z_5, \omega^9 z_6).$$

$g_i$	1	-1	$a$	$a^2$	$a^4$	$\pm b$	$\pm b^2$	$\pm b^3$	$-a$	$-a^2$	$-a^4$
$ C_G(g_i) $	52	52	13	13	13	4	4	4	13	13	13
$\phi_1$	4	-4	$\alpha$	$\beta$	$\gamma$	0	0	0	$-\alpha$	$-\beta$	$-\gamma$
$\phi_2$	4	-4	$\beta$	$\gamma$	$\alpha$	0	0	0	$-\beta$	$-\gamma$	$-\alpha$
$\phi_3$	4	-4	$\gamma$	$\alpha$	$\beta$	0	0	0	$-\gamma$	$-\alpha$	$-\beta$

Table 1: Some characters of irreducible representations for  $F_{13,4}$ .Figure 3: Isotropy subgroup lattice for  $\Gamma$  action on  $V_1, V_3, V_3$ 

We suppose that  $\mathbb{Z}_2$  acts as  $-I$  on those representations. The lattice of isotropy subgroups is given in Figure 3. Let  $\mu = x + iy$  and  $\nu = u + iv$ , with  $x, y, u, v \in \mathbb{R}$ . For  $j \in \{1, 2, 3\}$  we have

$$\text{Fix}_{V_j}(\mathbb{Z}_4(b)) = \{(x, 0, x, 0)\}, \quad \text{Fix}_{V_j}(\mathbb{Z}_4(-b)) = \{(x, 0, -x, 0)\},$$

and both fixed point subspaces have dimension one. Moreover,

$$\text{Fix}_{V_j}(\mathbb{Z}_2(b^2)) = \{(x, 0, u, 0)\}, \quad \text{Fix}_{V_j}(\mathbb{Z}_2(-b^2)) = \{(0, y, 0, v)\}$$

and so both fixed point subspaces have dimension two. Thus,  $\mathbb{Z}_2(-b^2)$  is a maximal isotropy subgroup with fixed point subspace of dimension 2.

Considering mappings of the type  $f : V_i \times V_j \rightarrow V_k$ , with  $i, j, k \in \{1, 2, 3\}$  then the indices of the isotropy subgroups are:

$$s(\mathbb{Z}_4(b)) = 1, \quad s(\mathbb{Z}_4(-b)) = 1, \quad s(\mathbb{Z}_2(b^2)) = 2, \quad s(\mathbb{Z}_2(-b^2)) = 2.$$

Letting  $f : V_1 \times V_2 \rightarrow V_2$  the lowest degree equivariant is the linear equivariant, since  $V_2$  is present in both the domain and range of  $f$ , this equivariant is a scalar multiple of the identity endomorphism from  $V_2$  to  $V_2$  by Schur's Lemma. Looking at Theorem 6.2, we have  $\delta = 1$  as the domain and image share one irreducible representation; also there is only one isotypic component in the range therefore the assumptions of Theorem 6.2 are satisfied and we can conclude that  $\mathcal{E}_G \subset \partial\mathcal{E}_\Sigma$  for the maximal isotropy subgroups  $\mathbb{Z}_4(b)$ ,  $\mathbb{Z}_4(-b)$  and  $\mathbb{Z}_2(-b^2)$ .

Now, using Theorem 6.2, we can finally prove our main result.

**Proof of Theorem 1.8** From Proposition 5.1, we neglect the trivial representations of  $V$  and  $W$  for studying the zero set near  $0 \in V$ . Decompose the study of  $f^{-1}(0)$  along isotypic components of  $W$ . That is, let  $f = (f_1, \dots, f_k)$  where  $f_i : V \rightarrow W_i$  and consider  $f_i^{-1}(0)$  for all  $i = 1, \dots, k$ . The representations  $V$  and  $W_i$  satisfy the hypothesis of Theorem 6.2, therefore for all maximal isotropy types  $(\Sigma)$  of  $G$  with  $s(\Sigma) > s(G)$ ,  $\mathcal{E}_G^i \subset \partial \mathcal{E}_\Sigma^i$  for all  $i = 1, \dots, k$ . By Proposition 5.4,  $\mathcal{E}_G \subset \partial \mathcal{E}_\Sigma$  for all the above mentioned isotropy types. Since  $f$  is  $G$ -transverse to  $0 \in W$  at  $0 \in V$ ,  $\text{graph}_f$  intersects transversally a stratum  $S_G$  of  $\mathcal{E}_G$ . Then it also intersects nontrivially and transversally a stratum of  $\mathcal{E}_\Sigma$  for all maximal isotropy types  $(\Sigma)$  of  $G$  with  $s(\Sigma) > s(G)$ . Hence,  $f^{-1}(0) = \text{graph}_f^{-1}(\mathcal{E})$  near  $0 \in V$  contains branches of zeros for all maximal isotropy subgroups  $\Sigma$  of  $G$  with  $s(\Sigma) > s(G)$  and these have dimension  $s(\Sigma)$  by Proposition 4.5. ■

## 6.2 Second case

We now turn to the case where  $\delta = 0$ . That is,  $V$  and  $W$  do not share irreducible representations and so the smallest degree of nonzero  $G$ -equivariant homogeneous generators is  $d \geq 2$ . Hence, we cannot use the implicit function theorem as in the previous case. Instead, we show below how a theorem of Buchner *et al* [3] can be used systematically to study the inclusion of  $\mathcal{E}_G$  into  $\partial \mathcal{E}_\Sigma$  when  $\Sigma$  is a maximal isotropy subgroup with positive index.

A homogeneous polynomial  $Q(x)$  is said to be *regular on its zero set* if the Jacobian matrix  $dQ(x)$  is surjective for all  $x \in Q^{-1}(0) \setminus \{0\}$ . Here is the required result.

**Theorem 6.4 (Buchner et al [3])** *Let  $k \geq 2$  be an integer. Suppose  $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is smooth,  $\Gamma$ -equivariant and  $g(0) = 0$ ,  $dg(0) = 0, \dots, d^{k-1}g(0) = 0$ . Let  $Q$  be the  $k$ -form associated to  $d^k g(0)$  and assume that  $Q$  is regular on its zero set. Then there are  $\Gamma$ -invariant neighborhoods  $U_1, U_2$  containing  $0 \in \mathbb{R}^n$  and a smooth  $\Gamma$ -equivariant diffeomorphism  $\phi : U_1 \rightarrow U_2$  such that*

$$\phi(Q^{-1}(0) \cap U_1) = g^{-1}(0) \cap U_2, \quad \phi(0) = 0 \quad \text{and} \quad d\phi(0) = \text{identity}.$$

The important assumption to satisfy in order to use Theorem 6.4 is the regularity of  $Q$  on its zero set. In fact we need regularity on the zero set only for certain isotropy subgroups.

**Lemma 6.5** *If  $Q^{-1}(0) \neq \{0\}$  for some maximal isotropy subgroup  $\Sigma$  with  $s(\Sigma) \leq 0$ , then  $Q$  is not regular on its zero set.*

**Proof:** This is obvious for  $s(\Sigma) < 0$  since  $\dim \text{Fix}_V(\Sigma)$  is the maximal rank of  $dQ$  restricted to  $\text{Fix}_V(\Sigma)$  and

$$\dim \text{Fix}_V(\Sigma) < \dim \text{Fix}_W(\Sigma).$$

If  $s(\Sigma) = 0$  and  $Q(x, \tau) = 0$  for some  $x \neq 0$ , then  $dQ|_{\text{Fix}_V(\Sigma)}$  has a nonzero kernel since the line through  $x$  consists of zeroes of  $Q$ . ■

We now illustrate the use of Theorem 6.4 with a few examples.

**Example 6.6** Consider the group  $G = \mathbb{D}_6$  acting on  $V = \mathbb{C}^2$  and  $W = \mathbb{C}$  as follows. Let  $(z_1, z_2) \in V$  and  $w \in W$ , then

$$\kappa.(z_1, z_2) = (\bar{z}_1, \bar{z}_2), \quad \sigma.(z_1, z_2) = (e^{i\pi/3} z_1, e^{i\pi/3} z_2)$$



and

$$\kappa.w = \overline{w}, \quad \sigma.w = e^{2i\pi/3}w.$$

The lattice of isotropy subgroups with indices is given in Figure 4. We see that both  $\mathbb{Z}_2(\kappa)$  and  $\mathbb{Z}_2(\kappa\sigma)$  have positive index so that the zero set near 0 may include points with those isotropy types. The universal zero map is given by

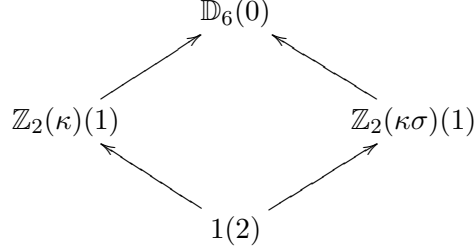


Figure 4: Lattice of isotropy types with index

$$F(z_1, z_2, t) = t_1 z_1^2 + t_2 z_1 z_2 + t_3 z_2^2 + t_4 \overline{z}_1^4 + t_5 \overline{z}_1^3 \overline{z}_2 + t_6 \overline{z}_1^2 \overline{z}_2^2 + t_7 \overline{z}_1 \overline{z}_2^3 + t_8 \overline{z}_2^4$$

where  $t \in \mathbb{R}^8$ . The mapping  $Q$  of lowest order equivariant terms is given by

$$Q(z_1, z_2, t_1, t_2, t_3) = t_1 z_1^2 + t_2 z_1 z_2 + t_3 z_2^2$$

and we look at the restriction of  $Q$  on the fixed point subspaces of  $\mathbb{Z}_2(\kappa)$  and  $\mathbb{Z}_2(\kappa\sigma)$ . We have

$$\text{Fix}_V(\mathbb{Z}_2(\kappa)) = \{(z_1, z_2) \mid z_1 = u \in \mathbb{R}, z_2 = v \in \mathbb{R}\}$$

so that  $Q^\Sigma = Q|_{\text{Fix}_V(\mathbb{Z}_2(\kappa))} = t_1 u^2 + t_2 uv + t_3 v^2$ . Thus,

$$dQ^\Sigma(u, v) = [2t_1 u + t_2 v, t_2 u + 2t_3 v].$$

We now show that  $Q$  is regular on its zero set for an open and dense set of values of  $t$ . The system  $dQ^\Sigma(u, v) = 0$  has a nonzero solution if and only if  $\Delta = 4t_1 t_3 - t_2^2 = 0$ . This means that  $Q$  is regular on its zero set for all values of  $(t_1, t_2, t_3)$  in the set  $\mathbb{R}^3 \setminus \Delta$ ; that is, for an open and dense set in  $\mathbb{R}^3$ . Now, we can choose  $(t_1^*, t_2^*, t_3^*) \in [\mathbb{R}^3 \setminus \Delta]$  such that  $t^* \in R_G$  and there exists  $(u, v) \neq (0, 0)$  such that  $Q^\Sigma(u, v, t_1^*, t_2^*, t_3^*) = 0$ . By homogeneity of  $Q^\Sigma$ , we know that  $Q^\Sigma(\lambda u, \lambda v, t_1^*, t_2^*, t_3^*) = 0$  for all  $\lambda \in \mathbb{R}$ ; that is, there is a line of zeros of  $Q^\Sigma$  through the origin. Now, consider  $F(u, v, t^*)$ . By Theorem 6.4,  $Q^{-1}(0)|_{\text{Fix}_V(\Sigma)}$  is homeomorphic to  $F^{-1}(0)|_{\text{Fix}_V(\Sigma)}$  near 0. Hence, there exists a sequence  $(x_n, t^*) \subset \mathcal{E}_\Sigma$  such that  $(x_n, t^*) \rightarrow (0, t^*)$ . That is,  $\mathcal{E}_G \subset \partial \mathcal{E}_{\mathbb{Z}_2(\kappa)}$ . A similar argument holds for the isotropy subgroup  $\mathbb{Z}_2(\kappa\sigma)$ . Thus, if a map in  $C_G^\infty(\mathbb{C}^2, \mathbb{C})$  is  $G$ -transverse to  $0 \in \mathbb{C}$  at  $0 \in \mathbb{C}^2$ , then the zero set in a neighborhood of the origin contains smooth 1-dimensional submanifolds of points with isotropy types  $\mathbb{Z}_2(\kappa)$  and  $\mathbb{Z}_2(\kappa\sigma)$ .

**Example 6.7** We return to the case of Example 6.3 and we now consider the case of  $C_G^\infty(V_1 \times V_2, V_3)$ . The polynomial equivariant of smallest degree is 3 since  $\delta = (\chi_{V_1 \times V_2}, \chi_{V_3}) = 0$  and there can be no quadratic equivariants because of the nontrivial action of  $-I$ . In order to apply

Theorem 6.4, we need to compute the cubic equivariants of the universal zero map  $F$ . The space of cubic equivariants for  $F$  is nine-dimensional and the mapping  $Q$  of cubic equivariants is

$$Q(z, t) = t_1 \begin{bmatrix} z_4^3 \\ \bar{z}_3^3 \end{bmatrix} + t_2 \begin{bmatrix} \bar{z}_3 \bar{z}_4^2 \\ z_3^2 \bar{z}_4 \end{bmatrix} + t_3 \begin{bmatrix} \bar{z}_2 z_3 \bar{z}_4 \\ z_1 z_3 z_4 \end{bmatrix} + t_4 \begin{bmatrix} z_2^2 z_3 \\ \bar{z}_1^2 z_4 \end{bmatrix} + t_5 \begin{bmatrix} \bar{z}_1 z_3 z_4 \\ \bar{z}_2 \bar{z}_3 z_4 \end{bmatrix} \\ + t_6 \begin{bmatrix} \bar{z}_1 z_2 \bar{z}_3 \\ \bar{z}_1 \bar{z}_2 \bar{z}_4 \end{bmatrix} + t_7 \begin{bmatrix} z_1 \bar{z}_3^2 \\ z_2 \bar{z}_4^2 \end{bmatrix} + t_8 \begin{bmatrix} z_1 \bar{z}_2 z_4 \\ z_1 z_2 \bar{z}_3 \end{bmatrix} + t_9 \begin{bmatrix} z_1^2 z_2 \\ \bar{z}_1 \bar{z}_2^2 \end{bmatrix}$$

Since all the maximal isotropy subgroups have positive indices, we check how the condition of regularity on its zero set is satisfied by  $Q$ . Let  $z_j = x_j + iy_j$ ,  $j = 1, 2, 3, 4$ , and  $\Sigma_1 = \mathbb{Z}_4(b)$ ,  $\Sigma_2 = \mathbb{Z}_4(-b)$ , and  $\Sigma_3 = \mathbb{Z}_2(-b_2)$  and letting  $Q^{\Sigma_i}$  denote again the cubic equivariants restricted to the respective fixed point subspaces. We have,

$$Q^{\Sigma_1}(x_1, x_3, t) = (t_1 + t_2)x_3^3 + (t_3 + t_5 + t_7)x_1x_3^2 + (t_4 + t_6 + t_8)x_1^2x_3 + t_9x_1^3$$

$$Q^{\Sigma_2}(x_1, x_3, t) = (-t_1 + t_2)x_3^3 + (t_3 - t_5 + t_7)x_1x_3^2 + (t_4 - t_6 + t_8)x_1^2x_3 - t_9x_1^3$$

and letting  $y = (y_1, y_2, y_3, y_4)$  we have  $(-i)Q^{\Sigma_3}(y, t)$

$$= t_1 \begin{bmatrix} -y_4^3 \\ y_3^3 \end{bmatrix} + t_2 \begin{bmatrix} y_3y_4^2 \\ -y_3^2y_4 \end{bmatrix} + t_3 \begin{bmatrix} -y_2y_3y_4 \\ -y_1y_3y_4 \end{bmatrix} + t_4 \begin{bmatrix} -y_2^2y_3 \\ y_1^2y_4 \end{bmatrix} + t_5 \begin{bmatrix} y_1y_3y_4 \\ -y_2y_3y_4 \end{bmatrix} \\ + t_6 \begin{bmatrix} -y_1y_2y_3 \\ y_1y_2y_4 \end{bmatrix} + t_7 \begin{bmatrix} -y_1y_3^2 \\ -y_2y_4^2 \end{bmatrix} + t_8 \begin{bmatrix} y_1y_2y_4 \\ y_1y_2y_3 \end{bmatrix} + t_9 \begin{bmatrix} -y_1^2y_2 \\ y_1y_2^2 \end{bmatrix}.$$

The linearization  $dQ^{\Sigma}$  for the maximal isotropy subgroups  $\Sigma_1$  and  $\Sigma_2$  are respectively

$$dQ^{\Sigma_1} = [(t_3 + t_5 + t_7)x_3^2 + 2(t_4 + t_6 + t_8)x_1x_3 + 3t_9x_1^2, 3(t_1 + t_2)x_3^2 + 2(t_3 + t_5 + t_7)x_1x_3 + (t_4 + t_6 + t_8)x_1^2]$$

and

$$dQ^{\Sigma_2} = [(t_3 - t_5 + t_7)x_3^2 + 2(t_4 - t_6 + t_8)x_1x_3 - 3t_9x_1^2, 3(-t_1 + t_2)x_3^2 + 2(t_3 - t_5 + t_7)x_1x_3 + (t_4 - t_6 + t_8)x_1^2]$$

These two cases can be analyzed as in Example 6.6 and one can show that  $\mathcal{E}_G \subset \partial\mathcal{E}_{\Sigma_1}$  and  $\mathcal{E}_G \subset \partial\mathcal{E}_{\Sigma_2}$ . We leave the details to the reader. We do the case of  $\Sigma_3$  since the details of the computations are slightly different. The linearization of  $Q^{\Sigma_3}$  is

$$dQ_3^{\Sigma_3}(y) = i[C_1(y), C_2(y), C_3(y), C_4(y)]$$

where

$$C_1(y) = \begin{bmatrix} t_5y_3y_4 - t_6y_2y_3 - t_7y_3^2 + t_8y_2y_4 - 2t_9y_1y_2 \\ -t_3y_3y_4 + 2t_4y_1y_4 + t_6y_2y_4 + t_8y_2y_4 + t_9y_2^2 \end{bmatrix} \\ C_2(y) = \begin{bmatrix} -t_3y_3y_4 - 2t_4y_2y_3 - t_6y_1y_3 + t_8y_1y_4 - t_9y_1^2 \\ -t_5y_3y_4 + t_6y_1y_4 - t_7y_4^2 + t_8y_1y_3 + 2t_9y_1y_2 \end{bmatrix}, \\ C_3(y) = \begin{bmatrix} t_2y_4^2 - t_3y_2y_4 - t_4y_2^2 - t_6y_1y_3 - 2t_7y_1y_3 - 2t_7y_1y_3 \\ 3t_1y_3^2 - 2t_2y_3y_4 - t_3y_1y_4 - t_5y_2y_4 + t_8y_1y_2 \end{bmatrix}, \\ C_4(y) = \begin{bmatrix} -3t_1y_4^2 + 2t_2y_3y_4 - t_3y_2y_3 + t_5y_1y_3 + t_8y_1y_2 \\ -t_2y_3^2 - t_3y_1y_4 + t_4y_1^2 - t_5y_2y_3 + t_6y_1y_2 - 2t_7y_2y_4 \end{bmatrix}.$$

We now show that  $Q^{\Sigma_3}$  is regular on its zero set for an open and dense set of values of  $(t_1, \dots, t_9) \in \mathbb{R}^9$ . Let  $M_{ij}(y)$  be the  $2 \times 2$  subdeterminants obtained with columns  $C_i$  and  $C_j$  where  $i < j$ . Consider the homogeneous mapping of degree four  $G : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  given by

$$G(y) = (M_{12}(y), M_{13}(y), M_{14}(y), M_{23}(y)).$$

From van der Waerden [20] Section 16.5, a set of  $n$  homogeneous polynomials in  $n$  variables with complex coefficients has a nontrivial common zero if and only if a resultant  $R$  in the coefficients of the homogeneous polynomials vanishes. The resultant  $R$  associated to  $G(y)$  is a function of  $(t_1, \dots, t_9)$ . Therefore,  $G(y) = 0$  for some nonzero  $y$  implies  $R(t_1, \dots, t_9) = 0$ . A straightforward calculation with a symbolic algebra package shows that the components of  $G$  are distinct and linearly independent, therefore,  $R(t_1, \dots, t_9) = 0$  is a proper algebraic subset of  $\mathbb{R}^9$ ; that is, the complement of  $R^{-1}(0)$  is an open and dense set. We choose  $(t_1, \dots, t_9) \in \mathbb{R}^9 \setminus R^{-1}(0)$  so that  $Q^{\Sigma_3}$  is necessarily regular on its zero set. We choose  $t \in R_G$  such that  $(t_1, \dots, t_9) \in \mathbb{R}^9 \setminus R^{-1}(0)$ . Since,  $s(\Sigma_3) > 0$  and  $Q^{\Sigma_3}$  has odd degree, then  $Q^{\Sigma_3}$  vanishes for some nonzero  $y^*$ . The remainder of the argument using Theorem 6.4 is similar to the one used in Example 6.6. Therefore,  $\mathcal{E}_G \subset \partial\mathcal{E}_{\Sigma_3}$  and we can conclude that if a map in  $C_G^\infty(V, W)$  is  $G$ -transverse to  $0 \in W$  at  $0 \in V$ , then the zero set in a neighborhood of the origin contains smooth 1-dimensional submanifolds of points with isotropy types  $\Sigma_1$  and  $\Sigma_2$  and smooth 2-dimensional submanifolds of points with isotropy type  $\Sigma_3$ .

### 6.3 Third Case

For the case  $0 < \delta < r$ , we use a Lyapunov-Schmidt type reduction procedure. Restrict  $W$  to a subspace  $W'$  which has  $\delta$  irreducible representations  $U$ . We decompose  $W = W' \oplus W'^\perp$  with respect to a  $G$ -invariant inner product and define

$$F^1 : V \times \mathbb{R}^l \rightarrow W' \quad F^2 : V \times \mathbb{R}^l \rightarrow W'^\perp.$$

The mapping  $F^1$  satisfies the hypothesis of Theorem 6.2 and so the implicit function theorem is applied. Let  $V = U \oplus V'$  where  $U$  is  $G$ -isomorphic to  $W'$ , then there exist neighborhoods of 0 in  $U$  and  $V'$  and a smooth  $G$ -equivariant mapping  $\phi : V' \rightarrow U$  such that  $F^1(\phi(v'), v') \equiv 0$ . Note that

$$d_{v'} F^1(\phi(v'), v')|_{v'=0} = d_u F^1(0, 0) d_{v'} \phi(0) + d_{v'} F^1(0, 0) = 0$$

where  $d_{v'} F^1(0, 0) = 0$  since  $V'$  and  $W'$  do not share any irreducible representations. Because  $d_u F^1(0, 0)$  is a nonzero matrix (consequence of Schur's lemma), then  $d_{v'} \phi(0) = 0$ . By Theorem 6.2  $\mathcal{E}_G \subset \partial\mathcal{E}_\Sigma$  for all maximal isotropy subgroups  $\Sigma$  with  $s(\Sigma; V, W') > 0$ .

We now substitute the solution of  $F^1$  into  $F^2$  and obtain a mapping

$$\tilde{F}^2 : V' \rightarrow W'^\perp$$

where  $d_{v'} F^2(0) = 0$  since  $d_{v'} \phi(0) = 0$ . Now,  $V'$  and  $W'^\perp$  do not share irreducible representations and so we need to study  $\tilde{F}^2$  using the method outlined in section 6.2. That is, for all maximal isotropy subgroups  $\Sigma$  with  $s(\Sigma; V', W'^\perp) > 0$  we need to find out if  $\mathcal{E}_G \subset \partial\mathcal{E}_\Sigma$  for the mapping  $\tilde{F}^2$ . For those  $\Sigma$  which do, then we have an inclusion  $\mathcal{E}_G$  in  $\partial\mathcal{E}_\Sigma$  for the full map  $F$ . The next example illustrates this approach.

**Example 6.8** Let  $V = V_3^2 \times V_2$  and  $W = V_2^2$ . Then  $W' = V_2$  and

$$F^1 : V \times \mathbb{R}^l \rightarrow V_2 \quad F^2 : V \times \mathbb{R}^l \rightarrow V_2.$$

Note that we have the same three maximal isotropy subgroups  $\Sigma_i$   $i = 1, 2, 3$  as in Example 6.3 and  $s(\Sigma_i) = 1$  for  $i = 1, 2$  and  $s(\Sigma_3) = 2$ .

Using the implicit function theorem, there exists a smooth  $G$ -equivariant mapping  $\phi : V_3^2 \rightarrow V_2$  near 0 which solves  $F^1 = 0$ . We substitute  $v_2 = \phi(u_3, v_3)$  in the second mapping where  $(u_3, v_3) \in V_3^2$  and  $v_2 \in V_2$ . Since  $F^1$  has no quadratic  $G$ -equivariant terms, it is easily shown by implicit differentiation that second derivatives of  $\phi$  evaluated at 0 vanish. Therefore, as expected, the lowest degree terms in  $F^2(v_1, v_3, \phi(u_3, v_3))$  are cubic. Let  $Q^2$  denote the cubic degree homogeneous truncation of  $F^2$ , then

$$Q^2(u_3, v_3, \phi(u_3, v_3)) = d_{v_2} F^2(0, 0, 0) \phi(u_3, v_3) + Q_3(u_3, v_3, 0).$$

where  $Q_3(u_3, v_3, v_2)$  denotes the cubic  $G$ -equivariant terms in  $F^2(u_3, v_3, v_2)$ . Note that the space of cubic  $G$ -equivariant mappings in  $C_G^\infty(V_3^2, V_2)$  is six-dimensional. Let  $u_3 = (z_1, z_2)$  and  $v_3 = (z_3, z_4)$  the generators are:

$$\begin{bmatrix} z_3^2 \bar{z}_4 \\ z_3 z_4^2 \end{bmatrix}, \begin{bmatrix} \bar{z}_2 z_3^2 \\ z_1 z_4^2 \end{bmatrix}, \begin{bmatrix} z_1 z_3 \bar{z}_4 \\ z_2 z_3 z_4 \end{bmatrix}, \begin{bmatrix} z_1 \bar{z}_2 z_3 \\ z_1 z_2 z_4 \end{bmatrix}, \begin{bmatrix} z_1^2 \bar{z}_4 \\ z_2^2 z_3 \end{bmatrix}, \begin{bmatrix} z_1^2 \bar{z}_2 \\ z_1 z_2^2 \end{bmatrix}.$$

Calculations similar to the ones done in Example 6.3 show that  $\mathcal{E}_G \subset \partial \mathcal{E}_{\Sigma_i}$  for every isotropy subgroup  $\Sigma_i$  since we have  $s(\Sigma_i; V_3^2, V_2) = 1$  for  $i = 1, 2$  and  $s(\Sigma_3; V_3^2, V_2) = 2$ .

## 7 More Questions

Question 1.7 in all its generality is more complicated than the problems we have investigated in this paper. The main problem is the characterization of the restriction of  $F$  to a subgroup. This problem is not well understood in general since hidden symmetries [10] and deficiencies [19] can modify the expected structure of the equivariant mapping on the fixed-point subspace. A positive answer to Question 1.7 leads to the following question.

**Question 7.1** We know that inclusion of strata is transitive. So that if  $\mathcal{E}_G \subset \partial \mathcal{E}_\Sigma$  and  $\mathcal{E}_\Sigma \subset \partial \mathcal{E}_{\Sigma'}$ , then  $\mathcal{E}_G \subset \partial \mathcal{E}_{\Sigma'}$ . If Question 1.7 has a positive answer, a necessary condition for these inclusions is that  $s(G) < s(\Sigma) < s(\Sigma')$ . Is it a sufficient condition?

Finally, we consider the case of continuous groups

**Example 7.2** The standard  $\mathbf{O}(2)$  action on  $V = W = \mathbb{C}$  is given by  $\kappa.z = \bar{z}$  and  $\theta.z = e^{i\theta}z$ . It is easy to check (see [12]) that  $F(z, t) = tz$ . There are two isotropy types:  $\mathbf{O}(2)$  and  $\mathbb{Z}_2(\kappa)$  with

$$\mathcal{E}_{\mathbf{O}(2)} = \{(z, t) | z = 0, t \neq 0\} \quad \text{and} \quad \mathcal{E}_{\mathbb{Z}_2(\kappa)} = \{(z, t) | z \neq 0, t = 0\}.$$

Here  $\dim \mathcal{E}_{\mathbf{O}(2)} < \dim \mathcal{E}_{\mathbb{Z}_2(\kappa)}$  since  $s(\mathbf{O}(2)) = s(\mathbb{Z}_2(\kappa)) = 0$ , but  $\mathcal{E}_{\mathbf{O}(2)} \not\subset \partial \mathcal{E}_{\mathbb{Z}_2(\kappa)}$ .

**Question 7.3** How does one study the inclusions  $\mathcal{E}_G$  into  $\partial \mathcal{E}_\Sigma$  when  $G$  is a continuous group. We see from Example 7.2 that the inequality of indices is not a necessary condition for inclusion. The dimension of the normalizer certainly plays a role here and the question of group orbits of zeros also arises.

Future work on this topic certainly includes the questions brought up above. More importantly, we would like to understand whether Theorem 6.2 can be extended to the cases  $\delta = 0$  and  $0 < \delta < r$  by using a general argument along the lines of the one provided for Example 6.6, Example 6.3 and Example 6.8. One possible obstacle to the generalization is to find out whether the mapping  $Q$  of lowest degree equivariant generators depends explicitly on all the variables parametrizing fixed point subspaces of maximal isotropy subgroups  $\Sigma$  with positive index. Let  $\{x_1, \dots, x_n\}$  be the coordinates of an orthogonal basis of  $\text{Fix}_V(\Sigma)$  and suppose that  $Q$  does not depend explicitly on  $x_1$ , then for  $x_1 \neq 0$ ,  $Q(x_1, 0, \dots, 0) = 0$  and automatically  $Q$  is not regular on its zero set for all values of  $t$ . Therefore, in order to use Theorem 6.4 in a general way, we need to gain an understanding of the lowest degree equivariant generators for  $C_G^\infty(V, W)$  from which we can find out whether the problem of non-regularity of  $Q$  on its zero set persists. The fact that Theorem 1.1 restricts the problem to  $V$  faithful may be crucial in pursuing this issue. There may also be other ways of generalizing Theorem 6.2 to the other cases of  $\delta$  without using Theorem 6.4 and this would be an interesting development.

## Acknowledgements

This research is partly supported by the Natural Sciences and Engineering Research Council of Canada in the form of a Discovery Grant (PLB) and an Undergraduate Student Research Award (MH).

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